

AN UPPER BOUND ON THE GROWTH OF MINIMAL GRAPHS

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ABSTRACT. We prove that solutions to the minimal surface equation over unbounded simply connected domains and having boundary values 0 have at most exponential growth.

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MSC: 49Q05

1. INTRODUCTION

Let D be an unbounded plane domain. In this paper we consider the boundary value problem for the minimal surface equation

$$(1.1) \quad \begin{cases} \operatorname{div} \frac{\nabla u}{\sqrt{1 + |\nabla u|^2}} = 0 & \text{and } u > 0 \text{ in } D \\ u = 0 & \text{on } \partial D \end{cases}$$

The positivity assumption on u is for convenience; for the growth of $|u(z)|$, the sets where $u > 0$ and $u < 0$ can be dealt with separately since there can be at most three such component solutions [10].

We shall use complex notation and study the constraints on growth of nontrivial solutions to (1.1) as determined by the maximum

$$M(r, u) = \max u(z),$$

where the max is taken over the values $|z| = r$ with $z \in D$.

Theorem 1. *Let D be a simply connected domain and u satisfy (1.1). With $M(r, u)$ as defined above, there exists a constant $C > 0$ such that for sufficiently large r ,*

$$(1.2) \quad M(r, u) \leq e^{Cr}.$$

Here and elsewhere, the constants C with or without subscripts may depend upon u . We may simplify D by replacing u by $u - c$ in the set $u > c$ so that the boundary can be made real analytic. In fact, since by [10] there can be at most 3 disjoint domains corresponding to solutions to (1.1), then for c large enough we may assume that the boundary of D is a single smooth unbounded Jordan arc.

Theorem 1 was conjectured in [13] where (1.2) was proved for domains contained in a half plane. The upper half of the horizontal catenoid can be used to show that this is sharp. Further observations with the catenoid were made in [4] and [5].

Earlier we also obtained lower bounds.

Theorem A. [6] [16] *Let u be as in (1.1). Then u has at least logarithmic growth. If D is simply connected, then*

$$\liminf_{r \rightarrow \infty} \frac{\log M(r, u)}{\log r} \geq 1/2.$$

There are a number of results regarding the growth constraints of solutions to (1.1) with restrictions on the geometry of D . Prototypes for these are given in [6, pp. 3390-3393]

Theorem B. [13] *Let u be as in (1.1) with D a simply connected domain contained in a half plane. Then*

$$M(r, u) \geq Cr$$

for some constant $C > 0$ and r sufficiently large.

Theorem C. [14] *Let u be as in (1.1) with D a simply connected domain bounded by a Jordan arc, and containing a sector $\{z : |\arg z| < \lambda/2\}$ with $\pi < \lambda < 2\pi$. Then*

$$\limsup_{r \rightarrow \infty} \frac{\log M(r, u)}{\log r} \leq \pi/\lambda.$$

In contrast with Theorem C we have

Theorem D. [12]. *Let u be as in (1.1) with D simply connected, and let $\Theta(r)$ be the angular measure of the set $D \cap \{|z| = r\}$. If $\limsup_{r \rightarrow \infty} \Theta(r) = \lambda \geq \pi$ then*

$$\limsup_{r \rightarrow \infty} \frac{\log M(r, u)}{\log r} \geq \pi/\lambda.$$

2. ISOTHERMAL COORDINATES

We shall make use of the parametrization of the surface given by u in isothermal coordinates using Weierstrass functions $(x(\zeta), y(\zeta), U(\zeta))$ with ζ in the right half plane H . Our notation will then be given by

$$(2.1) \quad f(\zeta) = x(\zeta) + iy(\zeta) \quad \zeta = \sigma + i\tau = \rho e^{i\varphi} \in H.$$

Then $f(\zeta)$ is univalent and harmonic, and since D is simply connected it can be written in the form

$$(2.2) \quad f(\zeta) = h(\zeta) + \overline{g(\zeta)}$$

where $h(\zeta)$ and $g(\zeta)$ are analytic in H and

$$(2.3) \quad |h'(\zeta)| > |g'(\zeta)|.$$

We dismiss the trivial case $g' \equiv 0$ so that

$$(2.4) \quad U(\zeta) = \pm 2\Re e i \int \sqrt{h'(\zeta)g'(\zeta)} d\zeta.$$

(cf. [2, §10.2]).

Now, $z = f(\zeta)$, $u(f(\zeta)) = U(\zeta)$ and $U(\zeta)$ is harmonic and positive in H and vanishes on ∂H . Thus, (cf. [11, p. 151]),

$$(2.5) \quad U(\zeta) = C \Re e \zeta,$$

where C is a positive constant. Then, (2.5) with (2.4) gives

$$g'(\zeta) = -\frac{C}{h'(\zeta)}.$$

By rescaling we may assume that

$$(2.6) \quad f(0) = 0 \quad U(\zeta) = 2\Re e \zeta \quad g'(\zeta) = -1/h'(\zeta)$$

and then the *analytic dilatation* $a(\zeta)$ satisfies

$$(2.7) \quad a(\zeta) = g'(\zeta)/h'(\zeta) = -1/h'(\zeta)^2.$$

Furthermore, from (2.3) we have, in particular, that

$$(2.8) \quad |h'(\zeta)| = 1/|g'(\zeta)| > 1.$$

We shall also need the *conjugate function*, given in D by (e.g. ([8, p. 62])

$$(2.9) \quad v(z) = \int -\frac{u_y}{\sqrt{1+|\nabla u|^2}} dx + \frac{u_x}{\sqrt{1+|\nabla u|^2}} dy.$$

As in (2.5), in the parameter half plane H we may take this as

$$(2.10) \quad V(\zeta) = 2\Im m \zeta.$$

3. PRELIMINARIES

Lemma 1. *With $h(\zeta)$ as in (2.2) and (2.8),*

$$(3.1) \quad \log |h'(\zeta)| = \frac{\sigma}{\pi} \int_{-\infty}^{\infty} \frac{\log |h'(is)| ds}{|(\zeta - is)^2} \quad \zeta \in H.$$

Proof. Since by (2.8) $\log |h'|$ is a positive harmonic function in H it can be represented (cf. [11, p. 149])

$$\log |h'(\zeta)| = \frac{\sigma}{\pi} \int_{-\infty}^{\infty} \frac{\log |h'(is)| ds}{|(\zeta - is)^2} + c \quad c \geq 0.$$

Furthermore, by [11, Theorem IV. 19.], $\log h'(\zeta)/\zeta \rightarrow c$ and $h''(\zeta)/h'(\zeta) \rightarrow c$ in any proper subsector of H . By [14, Lemma 2] it follows that $c = 0$.

□

For $0 < \alpha < \pi/2$, let

$$S_\alpha = \{\zeta : -\alpha \leq \arg \zeta \leq \alpha\}.$$

It follows from (3.1) that there exists $k > 0$ depending on u such that for ζ sufficiently large,

$$\log |h'(\zeta)| > k/|\zeta| \quad \zeta \in S_{\pi/4},$$

so that

$$(3.2) \quad |h'(\zeta)| > \exp(k/|\zeta|) > 1 + k/|\zeta| \quad \zeta \in S_{\pi/4}.$$

Now, referring to the language of univalent harmonic mappings in the unit disk U as in [2], we can say [2, pp.78-79] that for an arbitrary point $\zeta_0 \in S_{\pi/4}$ and $f_0(\zeta) = f(\zeta) - f(\zeta_0)$,

$$(3.3) \quad F_1(z) = \frac{f_0(\eta z + \zeta_0)}{\eta h'(\zeta_0)} \in S_H \quad (\eta = |\zeta_0|/20, z \in U)$$

and

$$(3.4) \quad F_2(z) = \left(F_1(z) + \frac{1}{|h'(\zeta_0)|^2} \overline{F_1(z)} \right) \left(1 - \frac{1}{|h'(\zeta_0)|^4} \right)^{-1} \in S_H^0 \quad (z \in U).$$

4. A THEOREM OF CLUNIE AND SHEIL-SMALL

In this section, we shall apply the following lemma of J. Clunie and T. Sheil-Small [2, p.95] to (3.4).

Theorem E. *If $F \in S_H^0$, then $F(U)$ contains the disk $\{|w| < 1/16\}$.*

From Theorem E we have

Lemma 2. *Let u be as in Theorem 1, and f as in §2. Let $\zeta_0 \in S_{\pi/4}$ be such that (3.2) holds, $w_0 = f(\zeta_0)$, and η as in (3.3). Then there exists $C > 0$ such that the image of the disk $|\zeta - \zeta_0| < \eta$ covers the disk $|w - w_0| < C$.*

Proof. With $\zeta = \eta e^{i\varphi}$ in (3.3) and (3.4),

$$\begin{aligned} \frac{f(\zeta) - w_0}{\eta h'(\zeta_0)} + \frac{\overline{f(\zeta) - w_0}}{|h'(\zeta_0)|^2 \eta \overline{h'(\zeta_0)}} &= \frac{f(\eta e^{i\varphi} + \zeta_0) - f(\zeta_0)}{\eta h'(\zeta_0)} + \frac{\overline{f(\eta e^{i\varphi} + \zeta_0) - f(\zeta_0)}}{|h'(\zeta_0)|^2 \eta \overline{h'(\zeta_0)}} \\ &= F_2(e^{i\varphi}) \left(1 - \frac{1}{|h'(\zeta_0)|^4} \right) = \left(1 - \frac{1}{|h'(\zeta_0)|^4} \right) w \end{aligned}$$

with $|w| \geq 1/16$.

Thus,

$$(4.1) \quad f(\zeta) - w_0 = \eta h'(\zeta_0) \left(1 - \frac{1}{|h'(\zeta_0)|^4} \right) w - \overline{\left(\frac{f(\zeta) - w_0}{h'(\zeta_0)^2} \right)}$$

Since $\eta = |\zeta_0|/20$, it follows from (3.2) that the magnitude of the first term on the right side of (4.1) is bounded below by some constant C independent of $\zeta_0 \in S_{\pi/4}$ and φ . This implies that there exists $C_1 > 0$ such that $|f(\zeta_1) - w_0| > C_1$. Indeed, if $|f(\zeta) - w_0|$ were less than $C/10$, then the right hand side of (4.1) would be greater than $9C/10$ which is a contradiction. \square

5. PROOF OF THE THEOREM

For our purposes we shall use the following consequence of Lemma 2. It follows from Lemma 2 that for $\zeta \in S_{\pi/4} \cap \{\Re \zeta = \sigma_0\}$, an increase of an amount $C_1 > 0$ in the value of $|f(\zeta)|$ can be achieved uniformly by an increase in $|\zeta|$ of at most $\eta = \sigma_0/20$ in $S_{\pi/4}$.

We begin with an arc $\gamma_0 = \{\Re \zeta = \sigma_0\} \cap S_{\pi/4}$ and let $\Gamma_0 = f(\gamma_0)$ with

$$r_0 = \min_{z \in \Gamma_0} |z|, \quad R_0 = \max_{z \in \Gamma_0} |z|.$$

We now expand Γ_0 outward by taking $z \rightarrow z(1 + C_1/(2|z|))$, and let Γ_1 denote that portion for which $\gamma_1 = f^{-1}(\Gamma_1)$ is in $S_{\pi/4}$.

Since each point ζ in γ_0 is the center of a disk centered at ζ and radius $\eta = \zeta/20$ and whose image covers a disk of radius C_1 , it follows that if $\gamma_1 = f^{-1}(\Gamma_1)$ and the union of these disks centered on points of γ_0 cover Γ_1 .

Thus,

$$(5.1) \quad r_1 = \min_{z \in \Gamma_1} |z| \geq r_0 + C_1/2 \quad \text{and} \quad R_1 = \max_{z \in \Gamma_1} |z| \leq R_0 + C_1/2,$$

and

$$(5.2) \quad \max_{\zeta \in \gamma_1} \Re \zeta < \sigma_0(1 + 1/20).$$

If γ_1 extends all the way across $S_{\pi/4}$, then we take $\tilde{\gamma}_1 = \gamma_1$. Otherwise we extend Γ_1 radially outward until its inverse under f hits $\partial S_{\pi/4}$. We denote the extended curve by $\tilde{\Gamma}_1$ and its preimage under f by $\tilde{\gamma}_1$.

Now, (5.1) remains in force on $\tilde{\Gamma}_1$. However, to estimate (5.2) we first recall by (2.9) that the conjugate function $v(z)$ has $|\nabla v(z)| < 1$ so that on the extension of Γ_1 we have

$$(5.3) \quad |v(z)| < \pi R_1 \quad z \in \tilde{\Gamma}_1 \setminus \Gamma_1.$$

In order to use this to estimate $u(z)$ we observe that any endpoint ζ' of γ_1 inside $S_{\pi/4}$ is covered by a disk of radius $|\zeta''|/20$ centered at an endpoint ζ'' of γ_0'' , which is on $\partial S_{\pi/4}$. It follows that γ_1 extends across $S_{\pi/8}$ so that on the extension

$$|v(z')| > u(z') \tan(\pi/8) \quad z \in \tilde{\Gamma}_1 \setminus \Gamma_1.$$

Combining this with (5.3) we have

$$u(z) < \pi R_1 \cot(\pi/8) \quad z \in \tilde{\Gamma}_1 \setminus \Gamma_1.$$

Thus (5.2) becomes modified to

$$(5.4) \quad \max_{\zeta \in \tilde{\gamma}_1} \Re e \zeta < \sigma_0(1 + 1/20) + C_3(R_0 + C_1/2)$$

where $C_3 = \pi \cot(\pi/8)$.

We now repeat the process, expanding $\tilde{\Gamma}_1$ again by $z \rightarrow z(1 + C_1/(2|z|))$, and letting Γ_2 denote that portion for which $\gamma_2 = f^{-1}(\Gamma_2)$ is in $S_{\pi/4}$. We extend Γ_2 and γ_2 as before if need be and have

$$(5.5) \quad r_2 = \min_{z \in \tilde{\Gamma}_2} |z| \geq r_0 + C_1 \quad \text{and} \quad R_2 = \max_{z \in \tilde{\Gamma}_2} |z| \leq R_0 + C_1,$$

and

$$(5.6) \quad \max_{\zeta \in \tilde{\gamma}_2} \Re e \zeta < (\sigma_0(1 + 1/20) + C_3(R_0 + C_1/2))(1 + 1/20) + C_3(R_0 + C_1).$$

We continue this process obtaining sequences $\tilde{\gamma}_n$ and $\tilde{\Gamma}_n$ such that

$$(5.7) \quad r_n = \min_{z \in \tilde{\Gamma}_n} |z| \geq r_0 + nC_1/2,$$

$$(5.8) \quad R_n = \max_{z \in \tilde{\Gamma}_n} |z| \leq R_0 + nC_1/2,$$

and

$$(5.9) \quad \begin{aligned} \max_{\zeta \in \tilde{\gamma}_n} \Re e \zeta &< \sigma_0(1 + 1/20)^n + \sum_{k=1}^n C_3(R_0 + kC_1/2)(1 + 1/20)^{n-k} \\ &< \sigma_0(1 + 1/20)^n(1 + C_4n^2) \end{aligned}$$

It now follows readily from (2.5), (5.7) and (5.9) that

$$(5.10) \quad \log u(z) \leq C_5|z| + C_6 \log |z|. \quad (z \in \tilde{\Gamma}_n, n \rightarrow \infty).$$

We must now extend this to the boundary of D . For this we take the two endpoints $z_{n,l}$ and $z_{n,r}$ of $\tilde{\Gamma}_n$ and extend from $\tilde{\Gamma}_n$ on the circular arcs of radii $|z_{n,l}|$ and $|z_{n,r}|$ respectively to the boundary of D . Let Γ'_n denote this augmented arc.

We again use the conjugate function to estimate u on the circular arcs. At the points $z_{n,l}$ and $z_{n,r}$ we have $u = |v|$, and on the remainder of the circular arcs, $u < |v|$. Again,

by (2.9) we have $|\nabla v(z)| < 1$, so if z_n represents either of these endpoints, then on the circular arc from z_n to ∂D , by (5.8) we have $|v(z)| < \exp(C_5|z_n| + C_6 \log |z_n|) + \pi|z_n|$.

Putting this together with (5.8) and (5.10) we then have

$$(5.11) \quad \log u(z) \leq C_7|z| \quad (z \in \Gamma'_n, n \rightarrow \infty).$$

Given the "density" of the sequence $\{r_n\}$ we may interpolate the values of r in between giving (1.1). \square

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